REGULAR ELEMENTS DETERMINED BY GENERALIZED INVERSES

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Abstract. In a semiprime ring, von Neumann regular elements are determined by their inner inverses. In particular, for elements a, b of a von Neumann regular ring R, $a = b$ if and only if $I(a) = I(b)$, where $I(x)$ denotes the set of inner inverses of $x \in R$. We also prove that, in a semiprime ring, the same is true for reflexive inverses.

1. Introduction and preliminaries

In this short note, R will stand for an associative ring with unity. An element $a \in R$ is (von Neumann) regular if there exists $x \in R$ such that $a = axa$. Such an element x is called an inner inverse (also called von Neumann inverse or generalized inverse) of a. The set of regular elements of a ring R is denoted by $Reg(R)$. A ring R is regular if $Reg(R) = R$. Note that a regular ring is semiprime. In general, a regular element may have more than one inner inverse. We denote the set of inner inverses of a by $I(a)$. An element $x \in R$ is called an outer inverse of a if $xax = x$. Note that if $x \in I(a)$ then xax is both an inner and an outer inverse of a. An element $x \in R$ is called a reflexive inverse of a if it is both an inner and an outer inverse of a. Denote the set of reflexive inverses of a by $Ref(a)$. We first obtain a necessary and sufficient condition for $I(a) \subseteq I(b)$ (Lemma 6) and use this to prove that in a semiprime ring, for $a, b \in Reg(R)$, $I(a) = I(b)$ if and only if $a = b$ if and only if $Ref(a) = Ref(b)$. (Theorem 7 and Theorem 10).

We begin with a few key lemmas.

The following is well-known (cf. [1] Corollary 1, Chapter 2. p. 40.)

Lemma 1. *For $a \in R$ and $a_0 \in I(a)$, we have $I(a) = \{a_0 + t - a_0ataa_0 \mid t \in R\}$.*

As usual, $l(a)$ and $r(a)$ denote respectively the left and right annihilator of an element $a \in R$. We define the inner annihilator of an element $a \in R$, as $\{x \in R \mid axa = 0\}$ and denote it by $Iann(a)$.

The next Proposition gives a link between $I(a)$ and $Ref(a)$.

Proposition 2. *For $a \in Reg(R)$, let $\varphi_a : I(a) \longrightarrow Ref(a)$ be such that $\varphi_a(x) = xax$. Then*

1. *The map φ_a is onto.*
2. *$Ref(a) = I(a)aI(a)$.*
3. *If $x, y \in I(a)$ are such that $\varphi_a(x) = \varphi_a(y)$ then $x - y \in l(a) \cap r(a)$.*
4. *Let $x \in Ref(a)$, then $\varphi_a(x) = x$.*

Proof. This is clear. □

The next lemma is straightforward.

Lemma 3. *Let $a \in Reg(R)$ and $a_0 \in I(a)$. Write $e = aa_0$, $f = a_0a$ and $e' = 1-e$, $f' = 1-f$. Then*

(i) $Iann(a) = l(a) + r(a) = Re' + f'R.$
(ii) $I(a) = a_0 + Iann(a) = a_0 + Re' + f'R.$
(iii) *If* $a_0 \in Ref(a)$, *then* $Ref(a) = a_0 + fRe' + f'Re + f'RaRe'.$

Proof. We simply mention that the statement (iii) can be proved by using statement (2) of the above proposition 2. □

2. Characterization of $I(a)$ and $Ref(a)$

We can now state our first result in the following proposition.

Proposition 4. *Let R be a semiprime ring. If $a \in Reg(R)$, then for any $b \in R$, $bI(a)b$ is a singleton set if and only if $b \in Ra \cap aR$.*

Proof. Firstly, suppose that there exist $x, y \in R$ such that $b = xa = ay$ and let $a_0 \in I(a)$. We then have that, for any $t \in R$, $b(a_0 + t - a_0ataa_0)b = (xaa_0 + xat - xataa_0)ay = xay + xatay - xatay = xay$. This shows that indeed $bI(a)b$ is a singleton set.
Conversely, suppose that $bI(a)b = \{ba_0b\}$. We then have $b(a_0+t-a_0ataa_0)b = ba_0b$, for any $t \in R$. This implies that, for any $t \in R$, we have

$$b(t - a_0ataa_0)b = 0.$$

Substituting $(1-a_0a)t$ for t in this equality leads to $b(1-a_0a)tb = 0$, for any $t \in R$. The semiprimeness of R then implies that $b(1 - a_0a) = 0$, i.e. $b = ba_0a \in Ra$. Similarly, substituting t by $t(1 - aa_0)$ in the above equality gives $b = aa_0b$. In particular, $b \in aR$. □

We recall the following result obtained by S.K. Jain and M. Prasad ([2]).

Lemma 5. *Let R be a ring and let $b, d \in R$ such that $b + d$ is a Von Neumann regular element. Then the following are equivalent:*

(1) $bR \oplus dR = (b + d)R.$
(2) $Rb \oplus Rd = R(b + d).$
(3) $bR \cap dR = \{0\}$ *and* $Rb \cap Rd = \{0\}.$

The next proposition provides necessary and sufficient conditions as to when $I(a) \subseteq I(b)$, where $a, b \in Reg(R)$ and R is semiprime.

Proposition 6. *Let R be a semiprime ring and let $a, b \in Reg(R)$. Then $I(a) \subseteq I(b)$ if and only if $bR \cap dR = 0$ and $Rb \cap Rd = 0$ where $a = d + b$.*

Proof. Since $I(a) \subseteq I(b)$, we have $bxb = b$ for every $x \in I(a)$. By Proposition 4, $b \in Ra \cap aR$. Write $b = \alpha a = a\beta$ for some $\alpha, \beta \in R$. Then $bI(a)a = b$. Next, $bI(a)d = bI(a)a - bI(a)b = b - bI(a)b = 0$. Consider now $dI(a)b = aI(a)b - bI(a)b = a\beta - bI(a)b = b - b = 0$. We thus have

$$bI(a)d = 0 \quad \text{and} \quad dI(a)b = 0 \qquad (1)$$

Then, for any $x \in I(a)$, we have $b+d = a = axa = (b+d)x(b+d) = bxa+dxb+dxd = b + 0 + dxd$. This yields,

$$dI(a)d = d \qquad (2)$$

Now, we proceed to show $dR \cap bR = 0$. Let $bx = dy \in bR \cap dR$. Multiplying both sides of the equality (2) by y on the right and using $bx = dy$ we obtain $dI(a)bx = dy$. As proved above, we have $dI(a)b = 0$. and so $dy = 0$. This proves our assertion. Similarly, we show that $Rb \cap Rd = 0$. Let $xb = yd \in Rb \cap Rd$. Now, multiplying both sides of the equality (2) on the left by y, we get $ydI(a)d = yd$. This proves $xbI(a)d = yd$. Since $bI(a)d = 0$, we obtain $yd = 0$, proving $Rb \cap Rd = 0$.
The converse is easy using the above lemma 5. □

Next, we show, in particular, that the regular elements of a semiprime ring are equal if their sets of inner inverses are the same.

Theorem 7. *Let R be a semiprime ring and $a, b \in Reg(R)$. Then $I(a) = I(b)$ if and only if $a = b$.*

Proof. We only need to prove the sufficency. So assume that $I(a) = I(b)$. Proposotion 6 implies that we can write $a = b + d$ with $bR \cap dR = 0$, $Rd \cap Rb = 0$. Lemma 5 then gives that $(b+d)R = bR \oplus dR$. Since $I(a) = I(b)$ we also have $aI(b)a = \{a\}$ and $bI(a)b = \{b\}$ and Proposition 4 implies that $Ra = Rb$ and $aR = bR$. This leads to $aR = (b+d)R = bR \oplus dR = aR \oplus dR$. This forces d to be zero and hence $a = b$, as desired.

Alternatively we may invoke Hartwig's result (cf. [3]) in place of Lemma 5. This was pointed out to us by T.Y. Lam. Indeed, by our Proposition 4 we have $aR = bR$ and $Ra = Rb$, and thus by invoking Hartwig's result, there exist units $u, v \in R$ such that $b = au = va$. If $x \in I(a) = I(b)$, then $axa = a$ and $bxb = b$. The last equality implies that $vaxau = au$ and hence $va = a$. Thus $b = a$. □

Corollary 8. *Let R be a regular ring. Then $I(a) = I(b)$ if and only if $a = b$.*

Remark 9. Pace Nielsen remarked that, in the above theorem, the semiprime hypothesis can be replaced by the assumption that $a - b$ is regular. So assume $I(a) = I(b)$, and $a - b \in Reg(R)$. As in our Proposition 4 we obtain

$$bt(1 - aa_0)b = 0, \qquad (1)$$

for any $t \in R$. If b_1 is a reflexive inverse of b, we obtain, for any t in R, $ataa_0b = (ab_1a)taa_0b = a(b_1bb_1)ataa_0b = ab_1(bb_1ataa_0b)$. Replace t by b_1at in (1) and obtain

$$ataa_0b = atb. \qquad (2)$$

For $z \in I(a - b)$ we have $bzb = bza + azb - aza + a - b$. Using this equality we compute

$$bzb = bzba_0b = (bza + azb - aza + a - b)a_0b = bzaa_0b + azb - azaa_0b + aa_0b - b.$$

Using formulae (1) and (2) we get $aa_0b = b$ so that $bR \subseteq aR$. Symmetric arguments leads to $aR = bR$ and $Ra = Rb$ and Hartwig's theorem finishes the proof.

3. Reflexive inverses for semiprime rings

We conclude by characterizing the equality of $Ref(a) = Ref(b)$, and obtain the analogue of Theorem 7 for reflexive inverses of semiprime rings.

Theorem 10. *Let R be a semiprime ring such that $a, b \in Reg(R)$. Then $Ref(a) = Ref(b)$ if and only if $a = b$.*

Proof. Let $a_0 \in Ref(a) = Ref(b)$. Since $a = 0$ if and only if $Ref(a) = 0$, we may assume that a and b are not zero. Since $bRef(a)b = bRef(b)b = b$ and $Ref(a) = I(a)aI(a)$, we have that, for any t in R,

$$b(a_0 + t - a_0ataa_0)a(a_0 + t - a_0ataa_0)b = b \qquad (1)$$

Replacing t by $(1 - a_0a)t$ and noting that $a(1 - a_0a) = 0$, we obtain successively $b(a_0a + (1 - a_0a)ta)(a_0 + (1 - a_0a)t)b = b$ and $b(a_0a + (1 - a_0a)ta)(a_0)b = b$ and so $ba_0b + b(1 - a_0a)taa_0b = b$. Since $ba_0b = b$ this gives $b(1 - a_0a)taa_0b = 0$ for all $t \in R$. This leads to

$$aa_0b(1 - a_0a)taa_0b(1 - a_0a) = 0 \quad \forall t \in R.$$

The semiprimeness of R implies that $aa_0b(1 - a_0a) = 0$. Left multiplying by $a_0 \in ref(a)$, we get that $a_0b(1 - a_0a) = 0$ and hence since $a_0 \in I(b)$ we conclude that $b(1-a_0a) = 0$. Therefore we obtain that $Rb \subseteq Ra$ and by symmetry $Ra \subseteq Rb$ and hence $Ra = Rb$. In the same way replacing t by $t(1 - aa_0)$ in (1), we obtain $aR = bR$. The Hartwig's Theorem then gives us that there exist invertible elements $u, v \in R$ such that $a = bu$ and $b = av$. The argument at the end of the proof of the semiprime case (cf. Theorem 7) proves the theorem. □

We now give an example of a ring, showing that without the semipriness hypothesis both of the above theorems are false.

Example 11. Consider the $\mathbb{F}_2$-algebra

$$R = \mathbb{F}_2\langle a, b, x \mid axa = a, bxb = b, xax = x, xbx = x, a^2 = b^2 = ab = x^2 = 0\rangle$$

This ring is finite and $\{a, b, x, ax, bx, xa, xb, axb, bxa\}$ is a basis of R as an $\mathbb{F}_2$-vector space. It is easy to determine that $r(a) = r(b)\langle a, b, ax, bx, axb, bxa\rangle$, $l(a) = l(b) = \langle a, b, xa, xb, axb, bxa\rangle$ $I(a) = x + R$, $I(b) = x + R$, $ref(a) = \{x\} = ref(b)$. Of course, $(RaR)^2 = 0$, showing that R is not semiprime.

The next corollary is a direct consequence of Theorems 7 and 10.

Corollary 12. *Let a, b be elements of a semiprime ring R. Then the following are equivalent:*

(1) $I(a) = I(b)$,
(2) $a = b$,
(3) $ref(a) = Ref(b)$.

Remark 13. We close with the following comment. The question of the equality of two elements in a regular ring that have the same set of inner inverses arose while the authors have been working on the question: if, for a regular self-injective ring R, $I(c) = I(a) + I(b)$, $a, b, c \in R$, is it true that c is unique? If not, obtain a complete solution for c. We will discuss that in another paper which is in progress.

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References

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